The Alternating-Direction Implicit Method
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Introduction

• ADI Method
  - Developed by Peaceman and Rachford in 1955
  - Alternate line iterations in x- and y-directions

• Benchmark
  - Jacobi (Point and Line Iterative)
  - Gauss-Seidel (Point and Line Iterative)
  - Successive Over Relaxation (Point and Line Iterative)
  - Alternating-Direction Implicit
Stability Analysis for Point Iterative Methods

\[
\frac{T_{i,j,n+1} - T_{i,j,n}}{\Delta t} = \frac{T_{i-1,j,n} - 2T_{i,j,n} + T_{i+1,j,n}}{(\Delta x)^2} + \frac{T_{i,j-1,n} - 2T_{i,j,n} + T_{i,j+1,n}}{(\Delta y)^2}
\]

\[
\rho = \frac{(\Delta x)^2}{\Delta t} = \frac{(\Delta y)^2}{\Delta t}
\]

\[
T_{i,j,n+1} = T_{i,j,n} + \frac{1}{\rho} [T_{i-1,j,n} + T_{i+1,j,n} + T_{i,j-1,n} + T_{i,j+1,n} - 4T_{i,j,n}]
\]

\[
\epsilon_{i,j,n+1} = \epsilon_{i,j,n} + \frac{1}{\rho} [\epsilon_{i-1,j,n} + \epsilon_{i+1,j,n} + \epsilon_{i,j-1,n} + \epsilon_{i,j+1,n} - 4\epsilon_{i,j,n}]
\]

\[
\epsilon_{i,j,n} = \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} A_{p,q,n} \cos \beta_p x \cos \beta_q y
\]

\[
\frac{A_{p,q,n+1}}{A_{p,q,n}} = 1 - \frac{4}{\rho} \left( \sin^2 \frac{\beta_p \Delta x}{2} + \sin^2 \frac{\beta_q \Delta y}{2} \right)
\]

\[
\rho \geq 4 \quad \Delta t \leq \frac{(\Delta x)^2}{4} \quad \Delta t \leq \frac{(\Delta y)^2}{4}
\]
Stability Analysis for Line Iterative and ADI Methods

\[ \frac{T_{i,j,2n+1} - T_{i,j,2n}}{\Delta t} = \frac{T_{i-1,j,2n+1} - 2T_{i,j,2n+1} + T_{i+1,j,2n+1}}{(\Delta x)^2} + \frac{T_{i,j-1,2n} - 2T_{i,j,2n} + T_{i,j+1,2n}}{(\Delta y)^2} \]

\[ \frac{T_{i,j,2n+2} - T_{i,j,2n}}{\Delta t} = \frac{T_{i-1,j,2n+1} - 2T_{i,j,2n+1} + T_{i+1,j,2n+1}}{(\Delta x)^2} + \frac{T_{i,j-1,2n+2} - 2T_{i,j,2n+2} + T_{i,j+1,2n+2}}{(\Delta y)^2} \]

\[ A_{p,q,2n+1} [2 \cos \beta_p \Delta x - (2 + \rho)] = A_{p,q,2n} [-2 \cos \beta_q \Delta y + (2 - \rho)] \]

\[ A_{p,q,2n+2} [2 \cos \beta_q \Delta y - (2 + \rho)] = A_{p,q,2n+1} [-2 \cos \beta_p \Delta x + (2 - \rho)] \]

Line Iterative, Unstable

\[ \frac{A_{p,q,2n+1}}{A_{p,q,2n}} = \frac{\rho - 4 \sin^2 \left( \frac{\beta_q \Delta y}{2} \right)}{\rho - 4 \sin^2 \left( \frac{\beta_p \Delta x}{2} \right)} \]

ADI, Unconditionally Stable

\[ \frac{A_{p,q,2n+2}}{A_{p,q,2n}} = \frac{\rho - 4 \sin^2 \left( \frac{\beta_p \Delta x}{2} \right)}{\rho + 4 \sin^2 \left( \frac{\beta_p \Delta x}{2} \right)} \times \frac{\rho - 4 \sin^2 \left( \frac{\beta_q \Delta y}{2} \right)}{\rho + 4 \sin^2 \left( \frac{\beta_q \Delta y}{2} \right)} \]
Example Problem

\[ \nabla^2 T = 0 \]
\[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = 0 \]

Boundary Conditions

\[ T\left(r, \theta = \frac{\pi}{2}\right) = 0 \]
\[ T\left(r, \theta = \pi\right) = 0 \]
\[ T\left(r = 0.22, \theta\right) = 0 \]
\[ T\left(r = 1, \theta\right) = \sin(2\theta) \]

Analytical Solution

\[ T\left(r, \theta\right) = ar^2 \sin(2\theta) + br^{-2} \]
\[ a = \frac{256}{257} \quad b = \frac{1}{257} \]
Benchmark of Methods

![Comparison of Methods]

- Jacobi
- Gauss-Seidel
- SOR (omega = 1.1)
- ADI

**Method 2**:
- Line Iterative

**Method 1**:
- Point Iterative


